

Some asymptotic formulae for Bessel process^{*}

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Abstract

We recover in part a recent result of [5] on the asymptotic behaviors for tail probabilities of first hitting times of Bessel process. Our proof is based on a weak convergence argument. The same reasoning enables us to derive the asymptotic behaviors for the tail probability of the time at which the global infimum of Bessel process is attained, and for expected values relative to local infima. In addition, we give another proof of the result of [5] with improvement of error estimates, which complements in the case of noninteger dimensions the asymptotic formulae by [15] for first hitting times of multidimensional Brownian motion.

1 Introduction

For every $\nu \in \mathbb{R}$ and $a > 0$, we denote by $P_a^{(\nu)}$ the law on the space $C([0, \infty); \mathbb{R})$ of real-valued continuous paths over $[0, \infty)$, induced by Bessel process with index ν (or dimension $\delta = 2(\nu + 1)$) starting from a . For every $b \geq 0$, we denote by τ_b the first hitting time to b :

$$\tau_b(\omega) := \inf\{t \geq 0; \omega(t) = b\}, \quad \omega \in C([0, \infty); \mathbb{R}).$$

In Hamana-Matsumoto [5], they have shown the following asymptotic formulae for the tail probability of τ_b in the case $b < a$: for every $\nu > 0$,

$$P_a^{(\nu)}(\infty > \tau_b > t) = \frac{1}{(2t)^\nu \Gamma(\nu + 1)} b^{2\nu} \left\{ 1 - \left(\frac{b}{a} \right)^{2\nu} \right\} + O(t^{-\nu-\varepsilon}), \quad (1.1)$$

$$P_a^{(-\nu)}(\tau_b > t) = \frac{1}{(2t)^\nu \Gamma(\nu + 1)} a^{2\nu} \left\{ 1 - \left(\frac{b}{a} \right)^{2\nu} \right\} + O(t^{-\nu-\varepsilon}) \quad (1.2)$$

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as $t \rightarrow \infty$ for any $\varepsilon \in (0, \nu/(\nu + 1))$. Here Γ is the gamma function. Their proof uses computational estimates.

One of the purposes of this paper is to give a different proof of these two formulae based on a weak convergence argument; while our proof does not give asymptotic estimates for remainder terms as in (1.1) and (1.2), we think that it is straightforward. The same reasoning also provides the asymptotic behaviors for the tail probability of the time at which the global infimum of Bessel process is attained, and for some expected values related to local infima.

We devote the latter half of the paper to another proof of (1.1) and (1.2). The proof is based on an identity for hitting distributions that is an immediate consequence of the strong Markov property of Bessel process. The identity differs from the one used in [5] and makes it possible to do more precise estimates; it will be shown that the remainder terms decay at rate $t^{-2\nu}$ for $0 < \nu < 1$ except $\nu = 1/2$, $(\log t)/t^2$ for $\nu = 1$, and $t^{-\nu-1}$ for $\nu > 1$ and $\nu = 1/2$. See Theorems 3.1 and 3.2 and Remark 3.2 below. In the case the dimension δ is noninteger, these asymptotics complement a result by van den Berg [15] that deals with first hitting times of Brownian motion to general compact sets in dimension greater than or equal to 3; repeated use of comparison of the first hitting time with the last exit time is the method employed there, which our argument is also different from. We also remark that we do not treat the case $\nu = 0$, for which we refer the reader to the detailed study [13] by Uchiyama, where obtained are asymptotics of hitting distributions in question as well as asymptotic estimates on density functions of first hitting times.

We organize this paper as follows: In Section 2 we first prove Theorem 2.1, which recovers principal terms in (1.1) and (1.2); we reduce the proof to showing that a given sequence of probability measures on $C([0, \infty); \mathbb{R})$ is weakly convergent. This argument also proves Proposition 2.3, which is then applied to derive in Theorem 2.2 the asymptotic behavior for the tail probability of the time Bessel process with positive index attains its global infimum, and those for expected values involving its local infima. In Section 3 we prove Theorems 3.1 and 3.2 that improve (1.1) and (1.2); we do this by using an identity for hitting distributions given in Lemma 3.1. Finally in the appendix, we prove auxiliary facts that are referred to in Sections 2 and 3.

In the sequel we write Ω for $C([0, \infty); \mathbb{R})$. We equip Ω with the topology of compact uniform convergence. Unless otherwise stated, R denotes the coordinate process on Ω : $R_t(\omega) := \omega(t)$, $\omega \in \Omega$, $t \geq 0$. We also set

$$\mathcal{F}_t := \sigma(R_s, s \leq t), \quad t \geq 0, \quad \text{and} \quad \mathcal{F} := \bigvee_{t \geq 0} \mathcal{F}_t.$$

For any $x, y \in \mathbb{R}$, we write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. Other notation will be introduced as needed.

2 Main results and proofs

Throughout the paper ν denotes a positive index. One of the objectives of this section is to give a proof of

Theorem 2.1. *It holds that for every $b \in [0, a)$,*

$$\begin{aligned} \text{(i)} \quad & \lim_{t \rightarrow \infty} t^\nu P_a^{(\nu)}(\infty > \tau_b > t) = \frac{1}{2^\nu \Gamma(\nu + 1)} b^{2\nu} \left\{ 1 - \left(\frac{b}{a} \right)^{2\nu} \right\}; \\ \text{(ii)} \quad & \lim_{t \rightarrow \infty} t^\nu P_a^{(-\nu)}(\tau_b > t) = \frac{1}{2^\nu \Gamma(\nu + 1)} a^{2\nu} \left\{ 1 - \left(\frac{b}{a} \right)^{2\nu} \right\}. \end{aligned}$$

We begin with stating two facts in propositions.

Proposition 2.1. *It holds that for every $t > 0$,*

$$P_a^{(\nu)}|_{\mathcal{F}_t} = \left(\frac{R_t}{a} \right)^{2\nu} P_a^{(-\nu)}|_{\mathcal{F}_t \cap \{t < \tau_0\}}. \quad (2.1)$$

Proposition 2.2. *Let $t > 0$. Then for any $A \in \mathcal{F}_t$, we have*

$$\lim_{s \rightarrow \infty} P_a^{(-\nu)}(A | \tau_0 > s) = P_a^{(\nu)}(A).$$

These two facts seem well known; to make the paper self-contained, we provide their proofs in Appendix. We deduce from Proposition 2.1 the following lemma, which plays a key role throughout this section.

Lemma 2.1. *For every $x > 0$, it holds that as $t \rightarrow \infty$,*

$$t^\nu E_x^{(\nu)} \left[\left(\frac{1}{R_t} \right)^{2\nu} \right] \rightarrow \frac{1}{2^\nu \Gamma(\nu + 1)}. \quad (2.2)$$

Here and below, $E_x^{(\nu)}$ denotes the expectation with respect to $P_x^{(\nu)}$.

Proof. By the absolute continuity relation Proposition 2.1, the expectation on the left-hand side of (2.2) is equal to $x^{-2\nu} P_x^{(-\nu)}(\tau_0 > t)$, and hence admits the representation

$$E_x^{(\nu)} \left[\left(\frac{1}{R_t} \right)^{2\nu} \right] = \frac{1}{2^\nu \Gamma(\nu)} \int_t^\infty \frac{ds}{s^{\nu+1}} \exp \left(-\frac{x^2}{2s} \right)$$

(see Remark A.1 in Appendix). The assertion readily follows from this identity. \square

One may also prove the lemma by using the explicit representation for the transition densities of Bessel process.

For each $t \geq 0$, we set

$$I_t \equiv I_t(R) := \inf_{0 \leq s \leq t} R_s.$$

We also write I_∞ for $\inf_{t \geq 0} R_t$. Recall that for every $x > 0$ and $0 \leq y \leq x$,

$$P_x^{(\nu)}(I_\infty > y) = 1 - \left(\frac{y}{x}\right)^{2\nu}. \quad (2.3)$$

As in [5], we also use the following identity:

Lemma 2.2. *For every $b \in [0, a)$ and $t > 0$, it holds that*

$$P_a^{(\nu)}(I_t > b) = 1 - \left(\frac{b}{a}\right)^{2\nu} + E_a^{(\nu)} \left[\left(\frac{b}{R_t}\right)^{2\nu}; I_t > b \right]. \quad (2.4)$$

Proof. By the Markov property of Bessel process and (2.3),

$$\begin{aligned} P_a^{(\nu)}(I_\infty > b \mid \mathcal{F}_t) &= P_x^{(\nu)}(y \wedge I_\infty > b) \big|_{(x,y)=(R_t,I_t)} \\ &= \mathbf{1}_{\{I_t > b\}} \left\{ 1 - \left(\frac{b}{R_t}\right)^{2\nu} \right\} \end{aligned}$$

$P_a^{(\nu)}$ -a.s. Taking the expectation on both sides leads to (2.4). \square

Since $P_a^{(\nu)}(\tau_b > t) = P_a^{(\nu)}(I_t > b)$ and $P_a^{(\nu)}(\tau_b = \infty) = P_a^{(\nu)}(I_\infty > b) = 1 - (b/a)^{2\nu}$ by (2.3), we have from (2.4)

$$P_a^{(\nu)}(\infty > \tau_b > t) = E_a^{(\nu)} \left[\left(\frac{b}{R_t}\right)^{2\nu}; I_t > b \right]. \quad (2.5)$$

We are ready to prove Theorem 2.1.

Proof of Theorem 2.1. (i) Fix arbitrarily a strictly increasing sequence $\{t_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \infty$. For each n , we define the probability measure \tilde{P}_n on Ω by

$$\tilde{P}_n(A) := \frac{E_a^{(\nu)}[(R_{t_n})^{-2\nu}; R_{t_n} \in A]}{E_a^{(\nu)}[(R_{t_n})^{-2\nu}]}, \quad A \in \mathcal{F},$$

where $R_t^{t_n} := R_{t \wedge t_n}$, $t \geq 0$.

First we show that $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ is tight. Fix $t > 0$ and take $A' \in \mathcal{F}_t$. If we let n be such that $t_n \geq t$, then by Propositions 2.1 and 2.2,

$$\begin{aligned} \tilde{P}_n(A') &= P_a^{(-\nu)}(A' \mid \tau_0 > t_n) \\ &\xrightarrow{n \rightarrow \infty} P_a^{(\nu)}(A'). \end{aligned} \quad (2.6)$$

This convergence for any $A' \in \mathcal{F}_t$ entails in particular that by regarding each \tilde{P}_n as being defined on the path space $\Omega_t = C([0, t]; \mathbb{R})$ equipped with the uniform norm topology, $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ is tight as a sequence of probability measures on Ω_t , which is equivalent to

$$\limsup_{\delta \downarrow 0} \sup_{n \in \mathbb{N}} \tilde{P}_n \left(\omega \in \Omega_t; \sup_{\substack{|u-v| \leq \delta \\ 0 \leq u, v \leq t}} |\omega(u) - \omega(v)| > \varepsilon \right) = 0 \quad (2.7)$$

for any $\varepsilon > 0$ (see, e.g., [6, Theorem 2.4.10]). It is then clear that, with Ω_t replaced by Ω , (2.7) holds for any $t > 0$ and $\varepsilon > 0$, and hence the tightness of $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ follows.

As $t > 0$ is arbitrary, the convergence (2.6) also implies that $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ converges to $P_a^{(\nu)}$ in the sense of finite-dimensional distributions. Consequently, $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ converges weakly to $P_a^{(\nu)}$. Since $P_a^{(\nu)}(I_\infty = b) = 0$ by (2.3), the weak convergence entails that

$$\tilde{P}_n(I_\infty > b) \xrightarrow{n \rightarrow \infty} P_a^{(\nu)}(I_\infty > b). \quad (2.8)$$

By the definition of \tilde{P}_n , the left-hand side of (2.8) is equal to

$$\frac{E_a^{(\nu)}[(R_{t_n})^{-2\nu}; I_{t_n} > b]}{E_a^{(\nu)}[(R_{t_n})^{-2\nu}]}.$$

As the sequence $\{t_n\}_{n \in \mathbb{N}}$ is arbitrarily taken, we now conclude that

$$\lim_{t \rightarrow \infty} \frac{E_a^{(\nu)}[(R_t)^{-2\nu}; I_t > b]}{E_a^{(\nu)}[(R_t)^{-2\nu}]} = P_a^{(\nu)}(I_\infty > b), \quad (2.9)$$

which proves (i) by Lemma 2.1, (2.3) and (2.5).

(ii) By Proposition 2.1 we have

$$P_a^{(-\nu)}(\tau_b > t) \equiv P_a^{(-\nu)}(I_t > b) = E_a^{(\nu)} \left[\left(\frac{a}{R_t} \right)^{2\nu}; I_t > b \right]. \quad (2.10)$$

The assertion follows from this and (2.9). \square

The same reasoning as the proof of Theorem 2.1 (i) also yields the

Proposition 2.3. *For any continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, we have*

$$\lim_{t \rightarrow \infty} t^\nu E_a^{(\nu)} [f(I_t)(R_t)^{-2\nu}] = \frac{2\nu}{2^\nu a^{2\nu} \Gamma(\nu + 1)} \int_0^a z^{2\nu-1} f(z) dz.$$

Proof. We keep the notation in the proof of Theorem 2.1 (i). Note that the mapping

$$\Omega \ni \omega \mapsto \inf_{t \geq 0} \{a \wedge (\omega(t) \vee 0)\} =: I^{a,+}(\omega)$$

is bounded and continuous, and that $I_\infty = I^{a,+}(R)$ $P_a^{(\nu)}$ -a.s. As $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ converges weakly to $P_a^{(\nu)}$, we have for any continuous function f on \mathbb{R} ,

$$\int_{\Omega} f(I^{a,+}(R)) d\tilde{P}_n \xrightarrow{n \rightarrow \infty} E_a^{(\nu)}[f(I_\infty)].$$

By the definition of \tilde{P}_n , the left-hand side is equal to

$$\frac{E_a^{(\nu)}[f(I_{t_n})(R_{t_n})^{-2\nu}]}{E_a^{(\nu)}[(R_{t_n})^{-2\nu}]}.$$

The rest of the proof proceeds in the same way as the proof of Theorem 2.1 (i). \square

As an application of this proposition, we may prove further the following asymptotic formulae: We set

$$\rho_\infty := \inf\{t \geq 0; R_t = I_\infty\};$$

as we will see in Proposition A.1 below, ρ_∞ is a.s. the unique time at which the global infimum I_∞ is attained.

Theorem 2.2. (i) *For any $0 \leq b \leq a$, it holds that*

$$\lim_{t \rightarrow \infty} t^\nu P_a^{(\nu)}(I_t - I_\infty > b) = \frac{2\nu}{2^\nu a^{2\nu} \Gamma(\nu + 1)} \int_b^a z^{2\nu-1} (z - b)^{2\nu} dz. \quad (2.11)$$

In particular,

$$\lim_{t \rightarrow \infty} t^\nu P_a^{(\nu)}(\rho_\infty > t) = \frac{a^{2\nu}}{2^{\nu+1} \Gamma(\nu + 1)}. \quad (2.12)$$

(ii) *For any continuous function $g : \mathbb{R} \rightarrow \mathbb{R}$, it holds that*

$$\lim_{t \rightarrow \infty} t^\nu \{E_a^{(\nu)}[g(I_\infty)] - E_a^{(\nu)}[g(I_t)]\} = \frac{2\nu}{2^\nu a^{2\nu} \Gamma(\nu + 1)} \int_0^a (a^{2\nu} - 2z^{2\nu}) z^{2\nu-1} g(z) dz.$$

Proof. (i) By the Markov property of Bessel process and by (2.3),

$$\begin{aligned} P_a^{(\nu)}(I_t - I_\infty > b) &= E_a^{(\nu)} \left[P_x^{(\nu)}(y - y \wedge I_\infty > b) \Big|_{(x,y)=(R_t,I_t)} \right] \\ &= E_a^{(\nu)} \left[\frac{(I_t - b)^{2\nu}}{(R_t)^{2\nu}}; I_t > b \right]. \end{aligned} \quad (2.13)$$

Taking $f(z) = \{(z - b) \vee 0\}^{2\nu}$ in Proposition 2.3 leads to (2.11). The latter equality (2.12) follows by taking $b = 0$ in (2.11); indeed, as seen in the proof of Proposition A.1, one has $P_a^{(\nu)}(\rho_\infty > t) = P_a^{(\nu)}(I_t > I_\infty)$.

(ii) Again by the Markov property,

$$E_a^{(\nu)}[g(I_\infty) | \mathcal{F}_t] = E_x^{(\nu)}[g(y \wedge I_\infty)] \Big|_{(x,y)=(R_t,I_t)} \quad P_a^{(\nu)}\text{-a.s.}$$

for every $t > 0$. By (2.3), the $P_x^{(\nu)}$ -expectation on the right-hand side is calculated as

$$g(y) + \frac{h(y)}{x^{2\nu}}, \quad h(y) := 2\nu \int_0^y z^{2\nu-1} g(z) dz - y^{2\nu} g(y).$$

Hence we have

$$E_a^{(\nu)}[g(I_\infty)] - E_a^{(\nu)}[g(I_t)] = E_a^{(\nu)} \left[\frac{h(I_t)}{(R_t)^{2\nu}} \right].$$

Taking $f = h$ in Proposition 2.3 concludes the proof. \square

We give a remark on Theorem 2.2 (ii).

Remark 2.1. (1) We may allow the function g to have the set of discontinuity with Lebesgue measure 0; in particular, taking $g = \mathbf{1}_{(b,\infty)}$ recovers Theorem 2.1 (i).

(2) For the function h defined in the proof, the process

$$g(I_t) + \frac{h(I_t)}{(R_t)^{2\nu}}, \quad t \geq 0,$$

is, by definition, an $\{\mathcal{F}_t\}$ -martingale under $P_a^{(\nu)}$, which may be associated with the so-called *Azéma-Yor martingales* (see [1]); in fact, $\{(R_t)^{-2\nu}; t \geq 0\}$ is an $\{\mathcal{F}_t\}$ -local martingale and $\sup_{0 \leq s \leq t} (R_s)^{-2\nu} = (I_t)^{-2\nu}$.

We may also relate (2.12) to Theorem 2.1 (ii) in the following manner:

Proof of (2.12) via Theorem 2.1 (ii). Note that by taking $b = 0$ in (2.13),

$$\begin{aligned} P_a^{(\nu)}(\rho_\infty > t) &= P_a^{(\nu)}(I_t > I_\infty) \\ &= E_a^{(\nu)} \left[\left(\frac{I_t}{R_t} \right)^{2\nu} \right]. \end{aligned}$$

By the absolute continuity relation Proposition 2.1 and by Fubini's theorem, this is rewritten as

$$\begin{aligned} E_a^{(-\nu)} \left[\left(\frac{I_t}{a} \right)^{2\nu}; t < \tau_0 \right] &= \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(I_t > z) dz \\ &= \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(\tau_z > t) dz. \end{aligned}$$

For every $z \in (0, a)$ we have by Theorem 2.1 (ii) and (2.10),

$$a^{2\nu} \geq \frac{P_a^{(-\nu)}(\tau_z > t)}{E_a^{(\nu)}[(R_t)^{-2\nu}]} \xrightarrow{t \rightarrow \infty} a^{2\nu} \left\{ 1 - \left(\frac{z}{a} \right)^{2\nu} \right\},$$

and hence the bounded convergence theorem yields

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{P_a^{(\nu)}(\rho_\infty > t)}{E_a^{(\nu)}[(R_t)^{-2\nu}]} &= 2\nu \int_0^a z^{2\nu-1} \left\{ 1 - \left(\frac{z}{a} \right)^{2\nu} \right\} dz \\ &= \frac{1}{2} a^{2\nu}. \end{aligned}$$

This shows (2.12) by Lemma 2.1. \square

We conclude this section with a remark on the above proof.

Remark 2.2. In the proof we have just observed the identity

$$P_a^{(\nu)}(\rho_\infty > t) = \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(\tau_z > t) dz, \quad (2.14)$$

which may easily be extended, thanks to the Markov property, to

$$P_a^{(\nu)}(A \cap \{\rho_\infty > t\}) = \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(A \cap \{\tau_z > t\}) dz$$

for any $A \in \mathcal{F}_t$. This relation shows that the process $\{R_t; 0 \leq t \leq \rho_\infty\}$ under $P_a^{(\nu)}$ is identical in law with $\{\xi_t; 0 \leq t \leq \tau_Z(\xi)\}$, where ξ is a Bessel process with index $-\nu$ starting from a and Z is a random variable independent of ξ and distributed as $(2\nu/a^{2\nu})z^{2\nu-1} dz$, $z \in (0, a)$. In the case $\nu = 1/2$, this partly recovers the path decomposition of 3-dimensional Bessel process due to D. Williams (e.g., [10, Theorem VI.3.11]). We also refer the reader to [3, Corollary 4.14] for identities as (2.14) in a general framework of diffusion processes.

3 Asymptotic estimates for remainders

Independently of the argument used in the previous section, we prove in this section the next two theorems, which give sharp asymptotics for remainders in (1.1) and (1.2). In the sequel we fix $0 \leq b < a$ and set

$$C_\nu := \frac{a^{2\nu} - b^{2\nu}}{2^\nu \Gamma(\nu + 1)}$$

for every positive ν . When $\nu < 1$, we also set

$$\kappa_\nu := \int_1^\infty \frac{(v+1)^{2\nu} - v^{2\nu}}{v^{\nu+1}} dv \in (0, \infty). \quad (3.1)$$

Theorem 3.1. *It holds that*

- (i) for $\nu < 1$, $\lim_{t \rightarrow \infty} t^{2\nu} \left(P_a^{(\nu)}(\infty > \tau_b > t) - \frac{b^{2\nu}}{a^{2\nu}} \frac{C_\nu}{t^\nu} \right) = \frac{b^{4\nu}}{(2a^2)^\nu \Gamma(\nu + 1)} C_\nu (1 - \nu \kappa_\nu);$
- (ii) for $\nu = 1$, $\lim_{t \rightarrow \infty} \frac{t^2}{\log t} \left(P_a^{(1)}(\infty > \tau_b > t) - \frac{b^2}{a^2} \frac{C_1}{t} \right) = -\frac{b^4}{a^2} C_1;$
- (iii) for $\nu > 1$,

$$-\infty < \liminf_{t \rightarrow \infty} t^{\nu+1} \left(P_a^{(\nu)}(\infty > \tau_b > t) - \frac{b^{2\nu}}{a^{2\nu}} \frac{C_\nu}{t^\nu} \right) \leq \limsup_{t \rightarrow \infty} t^{\nu+1} \left(P_a^{(\nu)}(\infty > \tau_b > t) - \frac{b^{2\nu}}{a^{2\nu}} \frac{C_\nu}{t^\nu} \right) < 0.$$

Theorem 3.2. *It holds that*

- (i) for $\nu < 1$, $\lim_{t \rightarrow \infty} t^{2\nu} \left(P_a^{(-\nu)}(\tau_b > t) - \frac{C_\nu}{t^\nu} \right) = \frac{b^{2\nu}}{2^\nu \Gamma(\nu + 1)} C_\nu (1 - \nu \kappa_\nu);$
- (ii) for $\nu = 1$, $\lim_{t \rightarrow \infty} \frac{t^2}{\log t} \left(P_a^{(-1)}(\tau_b > t) - \frac{C_1}{t} \right) = -b^2 C_1;$
- (iii) for $\nu > 1$,

$$-\infty < \liminf_{t \rightarrow \infty} t^{\nu+1} \left(P_a^{(-\nu)}(\tau_b > t) - \frac{C_\nu}{t^\nu} \right) \leq \limsup_{t \rightarrow \infty} t^{\nu+1} \left(P_a^{(-\nu)}(\tau_b > t) - \frac{C_\nu}{t^\nu} \right) < 0.$$

Notice that since

$$P_a^{(\nu)}(\infty > \tau_b > t) = \left(\frac{b}{a} \right)^{2\nu} P_a^{(-\nu)}(\tau_b > t) \quad (3.2)$$

by (2.5) and (2.10), it suffices to prove Theorem 3.2. The proof utilizes the following relation for hitting distributions:

Lemma 3.1. *It holds that for every $t > 0$,*

$$\begin{aligned} & P_a^{(-\nu)}(\tau_b > t) \\ &= \frac{1}{P_b^{(-\nu)}(\tau_0 \leq t)} \left\{ P_a^{(-\nu)}(\tau_0 > t) - P_b^{(-\nu)}(\tau_0 > t) - \int_{D_t} P_a^{(-\nu)}(\tau_b \in ds) P_b^{(-\nu)}(\tau_0 \in du) \right\}, \end{aligned}$$

where

$$D_t := \{(s, u) \in (0, \infty)^2; s + u > t, s \leq t, u \leq t\}.$$

Proof. By the strong Markov property,

$$P_a^{(-\nu)}(\tau_0 > t, \tau_b \leq t) = \int_0^t P_a^{(-\nu)}(\tau_b \in ds) P_b^{(-\nu)}(\tau_0 > t - s).$$

By the definition of D_t , we may rewrite the right-hand side as

$$\int_{D_t} P_a^{(-\nu)}(\tau_b \in ds) P_b^{(-\nu)}(\tau_0 \in du) + P_a^{(-\nu)}(\tau_b \leq t) P_b^{(-\nu)}(\tau_0 > t).$$

On the other hand, the left-hand side is equal to

$$P_a^{(-\nu)}(\tau_0 > t) - P_a^{(-\nu)}(\tau_0 > t, \tau_b > t) = P_a^{(-\nu)}(\tau_0 > t) - P_a^{(-\nu)}(\tau_b > t)$$

since $\tau_0 \geq \tau_b$ $P_a^{(-\nu)}$ -a.s. Combining these leads to the desired identity. \square

As a preparatory step to the proof of Theorem 3.2, we give another proof of Theorem 2.1 (ii) using Lemma 3.1. Set

$$I(t) := \frac{P_a^{(-\nu)}(\tau_0 > t) - P_b^{(-\nu)}(\tau_0 > t)}{P_b^{(-\nu)}(\tau_0 \leq t)}, \quad J(t) := \frac{\int_{D_t} P_a^{(-\nu)}(\tau_b \in ds) P_b^{(-\nu)}(\tau_0 \in du)}{P_b^{(-\nu)}(\tau_0 \leq t)}$$

so that

$$P_a^{(-\nu)}(\tau_b > t) = I(t) - J(t) \tag{3.3}$$

by Lemma 3.1. Since we have the expression

$$P_x^{(-\nu)}(\tau_0 > t) = \frac{x^{2\nu}}{2^\nu \Gamma(\nu)} \int_t^\infty \frac{ds}{s^{\nu+1}} \exp\left(-\frac{x^2}{2s}\right), \quad t > 0, \tag{3.4}$$

for every $x \geq 0$ (see Remark A.1), it is immediate that

$$\lim_{t \rightarrow \infty} t^\nu I(t) = C_\nu.$$

Therefore in order to prove Theorem 2.1 (ii), it suffices to show that

$$\lim_{t \rightarrow \infty} t^\nu J(t) = 0.$$

To this end, take $t, \lambda > 0$ in such a way that $t > \lambda$. Because of the inclusion

$$D_t \subset \{(s, u) \in (0, \infty)^2; t + \lambda \geq s + u > t\} \\ \cup \{(s, u) \in (0, \infty)^2; s + u > t + \lambda, s \leq t, u \leq t\},$$

we have

$$\begin{aligned} & P_b^{(-\nu)}(\tau_0 \leq t) J(t) \\ & \leq P_a^{(-\nu)}(t + \lambda \geq \tau_0 > t) + \int_\lambda^t P_b^{(-\nu)}(\tau_0 \in du) P_a^{(-\nu)}(t \geq \tau_b > t + \lambda - u) \\ & =: J_1(t; \lambda) + J_2(t; \lambda), \end{aligned} \tag{3.5}$$

where the expression of J_1 is due to the strong Markov property. By (3.4) we have

$$\begin{aligned} J_1(t; \lambda) &\leq \frac{a^{2\nu}}{2^\nu \Gamma(\nu)} \int_t^{t+\lambda} \frac{ds}{s^{\nu+1}} \\ &= \frac{a^{2\nu}}{2^\nu \Gamma(\nu+1)} \left(\frac{1}{t}\right)^\nu \left\{1 - \left(\frac{t}{t+\lambda}\right)^\nu\right\}. \end{aligned}$$

Since there exists a positive constant c such that $1 - x^\nu \leq c(1 - x)$ for all $0 \leq x \leq 1$, we obtain an estimate

$$J_1(t; \lambda) \leq c_1 \frac{\lambda}{t^{\nu+1}}. \quad (3.6)$$

Here as well as in what follows, every c_i denotes a positive constant dependent only on a, b and ν . As for J_2 , we use (3.4) to rewrite

$$J_2(t; \lambda) = \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} \int_\lambda^t \frac{du}{(t + \lambda - u)^{\nu+1}} \exp\left\{-\frac{b^2}{2(t + \lambda - u)}\right\} P_a^{(-\nu)}(t \geq \tau_b > u). \quad (3.7)$$

Since $\tau_b \leq \tau_0$ $P_a^{(-\nu)}$ -a.s.,

$$\begin{aligned} P_a^{(-\nu)}(t \geq \tau_b > u) &\leq P_a^{(-\nu)}(\tau_0 > u) \\ &\leq \frac{a^{2\nu}}{2^\nu \Gamma(\nu+1)} \frac{1}{u^\nu} \end{aligned}$$

by (3.4). We substitute this estimate into (3.7) to obtain a bound

$$J_2(t; \lambda) \leq c_2 \int_\lambda^t \frac{du}{u^\nu (t + \lambda - u)^{\nu+1}}. \quad (3.8)$$

We now fix $\varepsilon \in (0, 1)$ arbitrarily and let $\lambda = \varepsilon t$. Then by (3.6),

$$\limsup_{t \rightarrow \infty} t^\nu J_1(t; \varepsilon t) \leq c_1 \varepsilon.$$

On the other hand, by (3.8),

$$J_2(t; \varepsilon t) \leq \frac{c_2}{(\varepsilon t)^{\nu+1}} \int_{\varepsilon t}^t \frac{du}{u^\nu},$$

whence

$$\lim_{t \rightarrow \infty} t^\nu J_2(t; \varepsilon t) = 0.$$

Combining these with (3.5), we have

$$\limsup_{t \rightarrow \infty} t^\nu J(t) \leq c_1 \varepsilon.$$

This shows Theorem 2.1 (ii) as ε is arbitrary.

We proceed to the proof of Theorem 3.2. We begin with the following lemma:

Lemma 3.2. *One has for every $x \geq 0$ and $t > 0$,*

$$P_x^{(-\nu)}(\tau_0 > t) = \frac{x^{2\nu}}{(2t)^\nu \Gamma(\nu + 1)} \exp\left(-\frac{x^2}{2t}\right) + P_x^{(-\nu-1)}(\tau_0 > t).$$

Proof. By integration by parts,

$$\int_t^\infty \frac{ds}{s^{\nu+1}} \exp\left(-\frac{x^2}{2s}\right) = \frac{1}{\nu t^\nu} \exp\left(-\frac{x^2}{2t}\right) + \frac{x^2}{2\nu} \int_t^\infty \frac{ds}{s^{\nu+2}} \exp\left(-\frac{x^2}{2s}\right).$$

Plugging this expression into (3.4), we obtain the equality. \square

Using this lemma, we divide $I(t)$ into three parts:

$$I(t) = I_1(t) + I_2(t) + I_3(t),$$

where we set

$$\begin{aligned} I_1(t) &= \frac{1}{(2t)^\nu \Gamma(\nu + 1)} \left\{ a^{2\nu} \exp\left(-\frac{a^2}{2t}\right) - b^{2\nu} \exp\left(-\frac{b^2}{2t}\right) \right\}, \\ I_2(t) &= P_a^{(-\nu-1)}(\tau_0 > t) - P_b^{(-\nu-1)}(\tau_0 > t), \\ I_3(t) &= \frac{P_b^{(-\nu)}(\tau_0 > t)}{P_b^{(-\nu)}(\tau_0 \leq t)} \left\{ P_a^{(-\nu)}(\tau_0 > t) - P_b^{(-\nu)}(\tau_0 > t) \right\}. \end{aligned} \quad (3.9)$$

Using the fact that $(1 - e^{-x})/x \xrightarrow{x \rightarrow 0} 1$ for I_1 and (3.4) for I_2 and I_3 , we see that

$$\begin{aligned} I_1(t) &= \frac{C_\nu}{t^\nu} - \frac{a^{2\nu+2} - b^{2\nu+2}}{(2t)^{\nu+1} \Gamma(\nu + 1)} + o(t^{-\nu-1}), \\ I_2(t) &= \frac{C_{\nu+1}}{t^{\nu+1}} + o(t^{-\nu-1}), \\ I_3(t) &= \frac{b^{2\nu}}{2^\nu \Gamma(\nu + 1)} \cdot \frac{C_\nu}{t^{2\nu}} + o(t^{-2\nu}). \end{aligned}$$

We put together these asymptotics into a proposition.

Proposition 3.1. *It holds that as $t \rightarrow \infty$,*

$$\begin{aligned} \text{(i) for } \nu < 1, \quad I(t) &= \frac{C_\nu}{t^\nu} + \frac{b^{2\nu}}{2^\nu \Gamma(\nu + 1)} \cdot \frac{C_\nu}{t^{2\nu}} + o(t^{-2\nu}); \\ \text{(ii) for } \nu = 1, \quad I(t) &= \frac{C_1}{t} - \frac{C_1^2}{2t^2} + o(t^{-2}); \\ \text{(iii) for } \nu > 1, \quad I(t) &= \frac{C_\nu}{t^\nu} - \frac{\nu C_{\nu+1}}{t^{\nu+1}} + o(t^{-\nu-1}). \end{aligned}$$

Remark 3.1. It is easily deduced from (3.9) that the terms of o -symbol in (i), (ii) and (iii) can be sharpened by $O(1/t^{(3\nu) \wedge (\nu+1)})$, $O(1/t^3)$ and $O(1/t^{(2\nu) \wedge (\nu+2)})$, respectively.

As to $J(t)$ in the decomposition (3.3), we let $t, \lambda > 0$ be such that $t > \lambda$ and set

$$K(t; \lambda) := \int_{\lambda}^t P_b^{(-\nu)}(\tau_0 \in du) (I(t + \lambda - u) - I(t)).$$

Recall (3.5).

Lemma 3.3. *The following estimates hold true:*

$$P_b^{(-\nu)}(\tau_0 \leq \lambda)J(t) \leq J_1(t; \lambda) + K(t; \lambda), \quad (3.10)$$

$$P_b^{(-\nu)}(\tau_0 \leq \lambda)J(t) \geq K(t; \lambda) - \int_{\lambda}^t P_b^{(-\nu)}(\tau_0 \in du) J(t + \lambda - u). \quad (3.11)$$

Proof. By the definition of J_2 and (3.3),

$$\begin{aligned} J_2(t; \lambda) &= \int_{\lambda}^t P_b^{(-\nu)}(\tau_0 \in du) (I(t + \lambda - u) - J(t + \lambda - u) - I(t) + J(t)) \\ &\leq K(t; \lambda) + \left\{ P_b^{(-\nu)}(\tau_0 \leq t) - P_b^{(-\nu)}(\tau_0 \leq \lambda) \right\} J(t), \end{aligned}$$

where the inequality is due to the nonnegativity of $J(t + \lambda - u)$. Plugging this estimate into (3.5), we obtain (3.10). The lower estimate (3.11) is proved similarly since

$$P_b^{(-\nu)}(\tau_0 \leq t)J(t) \geq J_2(t; \lambda)$$

by the definition of J_2 . □

Set

$$c_3 := \sup_{t>0} t^{(2\nu)\wedge(\nu+1)} \left| I(t) - \frac{C_{\nu}}{t^{\nu}} \right|,$$

which is finite by Proposition 3.1. Then, by noting

$$\frac{C_{\nu}}{t^{\nu}} - \frac{c_3}{t^{(2\nu)\wedge(\nu+1)}} \leq I(t) \leq \frac{C_{\nu}}{t^{\nu}} + \frac{c_3}{t^{(2\nu)\wedge(\nu+1)}}$$

for all $t > 0$ and by (3.4), we have upper and lower bounds on $K(t; \lambda)$ as follows:

$$K(t; \lambda) \leq \frac{b^{2\nu}}{2^{\nu}\Gamma(\nu)} C_{\nu} K_1(t; \lambda) + \frac{b^{2\nu}}{2^{\nu}\Gamma(\nu)} c_3 K_2(t; \lambda) + \frac{c_3 P_b^{(-\nu)}(t \geq \tau_0 > \lambda)}{t^{(2\nu)\wedge(\nu+1)}}, \quad (3.12)$$

$$K(t; \lambda) \geq \frac{b^{2\nu}}{2^{\nu}\Gamma(\nu)} C_{\nu} \exp\left(-\frac{b^2}{2\lambda}\right) K_1(t; \lambda) - \frac{b^{2\nu}}{2^{\nu}\Gamma(\nu)} c_3 K_2(t; \lambda) - \frac{c_3 P_b^{(-\nu)}(t \geq \tau_0 > \lambda)}{t^{(2\nu)\wedge(\nu+1)}}, \quad (3.13)$$

where

$$K_1(t; \lambda) := \int_{\lambda}^t \frac{du}{u^{\nu+1}} \left\{ \frac{1}{(t + \lambda - u)^{\nu}} - \frac{1}{t^{\nu}} \right\}, \quad K_2(t; \lambda) := \int_{\lambda}^t \frac{du}{u^{\nu+1} (t + \lambda - u)^{(2\nu)\wedge(\nu+1)}}.$$

Lemma 3.4. (1) It holds that as $t \rightarrow \infty$,

$$K_1(t; \lambda) = \frac{1}{(t + \lambda)^{2\nu}} \int_1^{t/\lambda} \frac{(v + 1)^{2\nu} - v^{2\nu}}{v^{\nu+1}} dv + O(t^{-\nu-1}).$$

(2) It holds that

$$\limsup_{t \rightarrow \infty} t^{(2\nu) \wedge (\nu+1)} K_2(t; \lambda) \leq \frac{2}{\nu \lambda^\nu}.$$

Proof. (1) By Lemma A.1 in Appendix,

$$\begin{aligned} K_1(t; \lambda) &= \frac{1}{\lambda^\nu (t + \lambda)^{2\nu}} \int_\lambda^t \frac{(u + \lambda)^{2\nu}}{u^{\nu+1}} du - \frac{1}{\nu} \left(\frac{1}{\lambda^\nu} - \frac{1}{t^\nu} \right) \frac{1}{t^\nu} \\ &= \frac{1}{\lambda^\nu (t + \lambda)^{2\nu}} \int_\lambda^t \frac{(u + \lambda)^{2\nu} - u^{2\nu}}{u^{\nu+1}} du - \frac{t^\nu - \lambda^\nu}{\nu \lambda^\nu t^{2\nu}} \left\{ 1 - \left(\frac{t}{t + \lambda} \right)^{2\nu} \right\}. \end{aligned}$$

Since the second term in the last member is of order $O(t^{-\nu-1})$, the assertion follows by changing variables with $u = \lambda v$ in the integral of the first term.

(2) When $\nu < 1$, we have by Lemma A.1,

$$\begin{aligned} K_2(t; \lambda) &= \left\{ \frac{1}{\lambda(t + \lambda)} \right\}^{3\nu} \int_\lambda^t (u + \lambda)^{3\nu-1} \left(\frac{\lambda^{\nu+1}}{u^{\nu+1}} + \frac{\lambda^{2\nu}}{u^{2\nu}} \right) du \\ &\leq \frac{2}{\lambda^\nu (t + \lambda)^{3\nu}} \int_\lambda^t \frac{(u + \lambda)^{3\nu-1}}{u^{2\nu}} du, \end{aligned}$$

from which the assertion follows readily. Here for the inequality, we used the fact that $(\lambda/u)^{\nu+1} \leq (\lambda/u)^{2\nu}$ for $u \geq \lambda$ as $\nu < 1$. The case $\nu \geq 1$ can be proved similarly (in fact, the limit exists in both cases). \square

We are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. (i) In view of the decomposition (3.3) and Proposition 3.1 (i), what to show is that

$$\lim_{t \rightarrow \infty} t^{2\nu} J(t) = \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} C_\nu \kappa_\nu. \quad (3.14)$$

Fix $\lambda > 0$ arbitrarily. By (3.12), Lemma 3.4 and the definition (3.1) of κ_ν , we have

$$\limsup_{t \rightarrow \infty} t^{2\nu} K(t; \lambda) \leq \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} C_\nu \kappa_\nu + \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} c_3 \cdot \frac{2}{\nu \lambda^\nu} + c_3 P_b^{(-\nu)}(\tau_0 > \lambda).$$

By (3.10) and (3.6), we see that the above estimate is also valid with the left-hand side replaced by

$$P_b^{(-\nu)}(\tau_0 \leq \lambda) \cdot \limsup_{t \rightarrow \infty} t^{2\nu} J(t).$$

As λ is arbitrary, we obtain by letting $\lambda \rightarrow \infty$,

$$\limsup_{t \rightarrow \infty} t^{2\nu} J(t) \leq \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} C_\nu \kappa_\nu. \quad (3.15)$$

We may use this upper bound to estimate the second term on the right-hand side of (3.11) in such a way that

$$\int_\lambda^t P_b^{(-\nu)}(\tau_0 \in du) J(t + \lambda - u) \leq c_4 K_2(t; \lambda)$$

for some c_4 . Then by (3.13) and Lemma 3.4,

$$\begin{aligned} & P_b^{(-\nu)}(\tau_0 \leq \lambda) \cdot \liminf_{t \rightarrow \infty} t^{2\nu} J(t) \\ & \geq \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} C_\nu \exp\left(-\frac{b^2}{2\lambda}\right) \kappa_\nu - \left(\frac{b^{2\nu}}{2^\nu \Gamma(\nu)} c_3 + c_4\right) \frac{2}{\nu \lambda^\nu} - c_3 P_b^{(-\nu)}(\tau_0 > \lambda), \end{aligned}$$

and hence letting $\lambda \rightarrow \infty$ also yields

$$\liminf_{t \rightarrow \infty} t^{2\nu} J(t) \geq \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} C_\nu \kappa_\nu.$$

This together with (3.15), proves (3.14).

(ii) We show that

$$\lim_{t \rightarrow \infty} \frac{t^2}{\log t} J(t) = b^2 C_1. \quad (3.16)$$

By Lemma 3.4(1),

$$K_1(t; \lambda) = \frac{1}{(t + \lambda)^2} \left(2 \log \frac{t}{\lambda} - \frac{\lambda}{t} + 1 \right) + O(t^{-2}),$$

hence for any $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{t^2}{\log t} K_1(t; \lambda) = 2. \quad (3.17)$$

Therefore by (3.12) and Lemma 3.4(2),

$$\limsup_{t \rightarrow \infty} \frac{t^2}{\log t} K(t; \lambda) \leq \frac{b^2}{2} C_1 \times 2 = b^2 C_1,$$

which entails that by (3.10), (3.6) and arbitrariness of λ ,

$$\limsup_{t \rightarrow \infty} \frac{t^2}{\log t} J(t) \leq b^2 C_1. \quad (3.18)$$

By this estimate, we bound the second term on the right-hand side of (3.11) as

$$\begin{aligned} \int_{\lambda}^t P_b^{(-1)}(\tau_0 \in du) J(t + \lambda - u) &\leq c_5 \int_{\lambda}^t \frac{du}{u^2} \cdot \frac{\log(t + \lambda - u)}{(t + \lambda - u)^2} \\ &\leq c_5 \log t \cdot K_2(t; \lambda). \end{aligned}$$

Combining this with Lemma 3.4 (2) and (3.17), we see that

$$P_b^{(-1)}(\tau_0 \leq \lambda) \cdot \liminf_{t \rightarrow \infty} \frac{t^2}{\log t} J(t) \geq b^2 C_1 \exp\left(-\frac{b^2}{2\lambda}\right) - \frac{2c_5}{\lambda},$$

and hence that

$$\liminf_{t \rightarrow \infty} \frac{t^2}{\log t} J(t) \geq b^2 C_1. \quad (3.19)$$

By (3.18) and (3.19), we conclude (3.16).

(iii) It suffices to prove

$$\limsup_{t \rightarrow \infty} t^{\nu+1} J(t) < \infty \quad (3.20)$$

by (3.3) and Proposition 3.1 (iii). For each fixed $\lambda > 0$, $K_1(t; \lambda) = O(t^{-\nu-1})$ by Lemma 3.4 (1). Therefore by Lemma 3.4 (2) and (3.12),

$$\limsup_{t \rightarrow \infty} t^{\nu+1} K(t; \lambda) < \infty.$$

Combining this with (3.10) and (3.6) leads to

$$P_b^{(-\nu)}(\tau_0 \leq \lambda) \cdot \limsup_{t \rightarrow \infty} t^{\nu+1} J(t) < \infty,$$

and hence (3.20). The proof is complete. \square

Proof of Theorem 3.1. Theorem 3.2 combined with (3.2) shows the theorem. \square

We close this section with a remark on Theorem 3.1.

Remark 3.2. (1) In the case $\nu = 1/2$, namely the case that the dimension $\delta = 2(\nu + 1)$ is 3, the limit exhibited in Theorem 3.1 (i) is equal to 0 since

$$1 - \nu \kappa_{\nu} = 1 - \frac{1}{2} \int_1^{\infty} \frac{dv}{v^{3/2}} = 0,$$

which is consistent with the fact that

$$\begin{aligned} P_a^{(1/2)}(\infty > \tau_b > t) &= \frac{b}{a} \int_t^{\infty} \frac{a-b}{\sqrt{2\pi s^3}} \exp\left\{-\frac{(a-b)^2}{2s}\right\} ds \\ &= \frac{b}{a} \cdot \frac{C_{1/2}}{t^{1/2}} + O(t^{-3/2}). \end{aligned}$$

Such cancellation in asymptotic expansions resulting in the remainder $O(t^{-3/2})$ in dimension 3, is observed in a generality by [15, Proposition 2], where obtained are the asymptotic formulae for tail probabilities of hitting times of Brownian motion to general nonpolar compact sets in dimension greater than or equal to 3.

(2) On the other hand, when $\nu \neq 1/2$, we observe that

$$1 - \nu\kappa_\nu > 0 \quad \text{for } \nu < 1/2 \quad \text{and} \quad 1 - \nu\kappa_\nu < 0 \quad \text{for } \nu > 1/2, \quad (3.21)$$

that is, the cancellation as in the case $\nu = 1/2$ does not take place. To verify (3.21), we change variables with $v = x/(1 - x)$ in the definition (3.1) of κ_ν to rewrite

$$\kappa_\nu = \int_{1/2}^1 \frac{1 - x^{2\nu}}{x^{\nu+1}(1 - x)^{\nu+1}} dx,$$

which entails that κ_ν is (strictly) increasing in ν since for every $x \in (0, 1)$, both $1 - x^{2\nu}$ and $x^{-\nu}(1 - x)^{-\nu}$ are increasing in ν .

(3) In the case that δ is an integer greater than or equal to 4, the assertions (ii) and (iii) of Theorem 3.1 also agree with [15, Proposition 2], the starting point x of Brownian motion and a compact set K therein being taken respectively as $|x| = a$ and K the ball of radius b centered at the origin. We remark that in the case $\delta = 4$ (i.e., $\nu = 1$), that proposition further reveals that the remainder after the term of order $(\log t)/t^2$ is $O(t^{-2})$; it also indicates that the constant b^4 appearing in the limit in Theorem 3.1 (ii) arises from the square of the Newtonian capacity $8\pi^2 b^2$ of a ball of radius b in \mathbb{R}^4 .

(4) At least for the decay rates of remainders, namely $O(t^{-2\nu})$ for $\nu < 1$ ($\nu \neq 1/2$), $O((\log t)/t^2)$ for $\nu = 1$, and $O(t^{-\nu-1})$ for $\nu > 1$ and $\nu = 1/2$, they may also be deduced from a recent result by Uchiyama [14] that gives asymptotic estimates of the density function of τ_b which are valid uniformly in starting points a including the case $\nu = 0$ as well.

(5) By using an explicit representation for the distribution function of τ_b , it is shown in [4, Theorem 4.1 (3)] that in the case δ is an odd integer with $\delta \geq 3$, the remainder decays at rate $t^{-\nu-1}$.

(6) Although we do not give details here, we may also prove that in the case $\nu > 1$,

$$\liminf_{t \rightarrow \infty} t^{\nu+1} J(t) \geq \frac{b^{2\nu}}{2^\nu \Gamma(\nu)} \cdot \frac{a^2 - b^2}{2(\nu - 1)},$$

the second factor on the right-hand side being the expected value of τ_b under $P_a^{(-\nu)}$. By this estimate, Proposition 3.1 (iii) and (3.2), the upper limit in Theorem 3.1 (iii) is estimated from above by

$$- \left(\frac{b}{a} \right)^{2\nu} \left\{ \nu C_{\nu+1} + \frac{b^{2\nu}(a^2 - b^2)}{2^{\nu+1}(\nu - 1)\Gamma(\nu)} \right\}.$$

Appendix

We append proofs of auxiliary facts referred to in preceding sections.

A.1 Proof of Proposition 2.1

The relation (2.1) may be deduced from the fact that the infinitesimal generator of Bessel process with positive index ν is identical with that of Bessel process with the opposite index, h -transformed by the function $h(x) = x^{2\nu}$, $x > 0$, that is harmonic in the sense that

$$\frac{1}{2}h''(x) + \frac{-2\nu + 1}{2x}h'(x) = 0.$$

For the reader's convenience, we give a proof of the proposition by means of a time-change and the Cameron-Martin relation. We refer the reader to [16] for the absolute continuity relationship for Bessel processes with nonnegative indices, which can also be proved by the same argument as below.

Proof of Proposition 2.1. Fix $a > 0$ and set $b = \log a$. Let $B = \{B_t; t \geq 0\}$ be a one-dimensional Brownian motion starting from b . For each $\mu \in \mathbb{R}$, we denote by $B^{(\mu)}$ the Brownian motion with drift μ : $B_t^{(\mu)} = B_t + \mu t$, $t \geq 0$. Let X denote the coordinate process on $\Omega = C([0, \infty); \mathbb{R})$. We define two functionals A, α of X by

$$A_t(X) := \int_0^t e^{2X_s} ds, \quad \alpha_t(X) := \inf\{s \geq 0; A_s(X) > t\}, \quad t \geq 0,$$

where we set $\inf \emptyset = \infty$. By Lamperti's relation (see, e.g., [8, Section 3]), there exists a Bessel process $R^{(\mu)}$ with index μ starting from a such that

$$\exp B_t^{(\mu)} = R_{A_t(B^{(\mu)})}^{(\mu)}, \quad t \geq 0, \tag{A.1}$$

and hence by the definition of α ,

$$\exp B_{\alpha_t(B^{(\mu)})}^{(\mu)} = R_t^{(\mu)}, \quad t < \tau_0(R^{(\mu)}). \tag{A.2}$$

Recall that $\tau_0(R^{(\mu)}) = \infty$ a.s. for $\mu \geq 0$ while $\tau_0(R^{(\mu)}) < \infty$ a.s. for $\mu < 0$; in fact,

$$\tau_0(R^{(\mu)}) = A_\infty(B^{(\mu)}) \tag{A.3}$$

by (A.1). We now fix $t > 0$ and take $\Gamma \in \mathcal{F}_t = \sigma(X_s, s \leq t)$. Let $\nu > 0$. Then by (A.2),

$$P(R^{(\nu)} \in \Gamma) = P\left(\exp B_{\alpha_t(B^{(\nu)})}^{(\nu)} \in \Gamma\right). \tag{A.4}$$

By definition, $\alpha_t(X)$ is a stopping time for the coordinate process X . Therefore the Cameron-Martin formula entails that (A.4) is equal to

$$\begin{aligned} & e^{-2\nu b} E \left[\exp \left\{ 2\nu B_{\alpha_t(B^{(-\nu)})}^{(-\nu)} \right\}; \exp B_{\alpha_t(B^{(-\nu)})}^{(-\nu)} \in \Gamma, \alpha_t(B^{(-\nu)}) < \infty \right] \\ &= a^{-2\nu} E \left[(R_t^{(-\nu)})^{2\nu}; R^{(-\nu)} \in \Gamma, \tau_0(R^{(-\nu)}) > t \right]. \end{aligned}$$

Here for the second line, we used (A.2), and the equivalence between $\alpha_t(B^{(-\nu)}) < \infty$ and $\tau_0(R^{(-\nu)}) > t$ that follows from (A.3). The proof is complete. \square

Remark A.1. By (A.3) and Dufresne's identity (see, e.g., [8, Section 2]), it holds that under $P_a^{(-\nu)}$,

$$\tau_0(R) \stackrel{(d)}{=} \frac{a^2}{2\gamma_\nu}.$$

Here γ_ν is a gamma random variable with parameter ν . Therefore one may find that

$$\begin{aligned} E_a^{(\nu)} \left[\left(\frac{a}{R_t} \right)^{2\nu} \right] &= P_a^{(-\nu)}(\tau_0 > t) \\ &= P(\gamma_\nu < a^2/(2t)) \\ &= \frac{a^{2\nu}}{2^\nu \Gamma(\nu)} \int_t^\infty \frac{ds}{s^{\nu+1}} \exp \left(-\frac{a^2}{2s} \right), \end{aligned}$$

where the first equality follows from (2.1).

A.2 Proof of Proposition 2.2

The proposition asserts that Bessel process with a negative index conditioned to stay positive is nothing but Bessel process with the opposite index. This seems to be a well-known fact and to have been rediscovered by several authors, see e.g., [12, Section 7]; we also refer to [2] for the case of drifted Brownian motions with nonsingular drift coefficients. The case $\nu = 1/2$ goes back to Knight [7, Theorem 3.1]. Roynette, Yor et al. extensively studied limit laws of Brownian motion normalized by various kinds of weight processes other than $\mathbf{1}_{\{\tau_0 > t\}}$, referring to those studies as *penalisation problems*; see [11] and references therein, where usage of Scheffé's lemma we employ in the proof below is also found. For related studies concerning quasi-stationary distributions (Yaglom limits), refer to [9].

Proof of Proposition 2.2. Fix arbitrarily a sequence $\{t_n\}_{n \in \mathbb{N}}$ of positive reals such that $\lim_{n \rightarrow \infty} t_n = \infty$. Let $N \in \mathbb{N}$ be such that $t_n > t$ for all $n \geq N$. By Proposition 2.1 and the Markov property, we have for every $n \geq N$,

$$\begin{aligned} P_a^{(-\nu)}(A \mid \tau_0 > t_n) &= \frac{E_a^{(\nu)}[(R_{t_n})^{-2\nu}; A]}{E_a^{(\nu)}[(R_{t_n})^{-2\nu}]} \\ &= E_a^{(\nu)}[M_n; A], \end{aligned} \tag{A.5}$$

where we set

$$M_n = \frac{E_{R_t}^{(\nu)}[(R_{t_n-t})^{-2\nu}]}{E_a^{(\nu)}[(R_{t_n})^{-2\nu}]}.$$

By Lemma 2.1, $M_n \rightarrow 1$ a.s. as $n \rightarrow \infty$. Moreover, $M_n \geq 0$ a.s. and $E_a^{(\nu)}[M_n] = 1$ for all $n \geq N$. Hence by Scheffé's lemma, $E_a^{(\nu)}[|M_n - 1|] \xrightarrow{n \rightarrow \infty} 0$, which entails that (A.5) converges to $P_a^{(\nu)}(A)$ as $n \rightarrow \infty$. Therefore we arrive at the conclusion as $\{t_n\}_{n \in \mathbb{N}}$ is arbitrary. \square

A.3 Statements and proofs of Proposition A.1 and Lemma A.1

Proposition A.1. *Let $\nu > 0$ and $a > 0$. Under $P_a^{(\nu)}$, the time at which the Bessel process R attains its global infimum I_∞ is a.s. unique.*

Proof. Write $\mathcal{T} \equiv \mathcal{T}(R) = \{s \geq 0; R_s = I_\infty\}$. Set $\rho_\infty = \inf \mathcal{T}$ as in Section 2 and $\bar{\rho}_\infty = \sup \mathcal{T}$. Note that \mathcal{T} is compact a.s. since R is continuous and $\lim_{s \rightarrow \infty} R_s = \infty$ a.s. It then holds that

$$\{\rho_\infty > t\} = \{R_s > I_\infty \text{ for all } s \in [0, t]\} \quad \text{a.s.}, \quad (\text{A.6})$$

namely the indicator functions of these two events are equal a.s. Indeed, it is obvious that the left-hand event is included in the right-hand event; for converse inclusion, since $t \notin \mathcal{T}$ and \mathcal{T} is compact a.s., we have $t < \inf \mathcal{T} = \rho_\infty$ a.s. By continuity, the right-hand side of (A.6) is written as $\{\inf_{0 \leq s \leq t} R_s > I_\infty\}$, and hence we have

$$\{\rho_\infty > t\} = \{I_t > \inf_{s \geq t} R_s\} \quad \text{a.s.}$$

Therefore by the Markov property and (2.3),

$$P_a^{(\nu)}(\rho_\infty > t) = E_a^{(\nu)} \left[\left(\frac{I_t}{R_t} \right)^{2\nu} \right].$$

Similarly

$$\{\bar{\rho}_\infty < t\} = \{I_t < \inf_{s \geq t} R_s\} \quad \text{a.s.},$$

from which it also follows that

$$\begin{aligned} P_a^{(\nu)}(\bar{\rho}_\infty \geq t) &= P_a^{(\nu)}(I_t \geq \inf_{s \geq t} R_s) \\ &= E_a^{(\nu)} \left[\left(\frac{I_t}{R_t} \right)^{2\nu} \right]. \end{aligned}$$

By the dominated convergence theorem, the mapping $[0, \infty) \ni t \mapsto E_a^{(\nu)}[(I_t/R_t)^{2\nu}]$ is continuous. Combining these we see that ρ_∞ and $\bar{\rho}_\infty$ have the same distribution, which implies that $\rho_\infty = \bar{\rho}_\infty$ a.s. since $\rho_\infty \leq \bar{\rho}_\infty$. This ends the proof. \square

Lemma A.1. *Let $c > 0$ and $\alpha, \beta \in \mathbb{R}$. For all $x \geq c$ one has*

$$\int_c^x \frac{dy}{y^\alpha(x+c-y)^\beta} = \left\{ \frac{1}{c(x+c)} \right\}^{\alpha+\beta-1} \int_c^x (y+c)^{\alpha+\beta-2} \left(\frac{c^\alpha}{y^\alpha} + \frac{c^\beta}{y^\beta} \right) dy.$$

Proof. By changing variables with $y = x + c - z$, we see that the left-hand side of the claimed identity is symmetric with respect to α and β , namely

$$\int_c^x \frac{dz}{z^\beta(x+c-z)^\alpha} = \int_c^x \frac{dz}{z^\alpha(x+c-z)^\beta},$$

which entails that it is equal to

$$\int_c^{(x+c)/2} \left\{ \frac{1}{z^\alpha(x+c-z)^\beta} + \frac{1}{z^\beta(x+c-z)^\alpha} \right\} dz.$$

Changing variables with $z = c(x+c)/(y+c)$ leads to the conclusion. \square

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